Theorem:

If the set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in a vector space V_F be linearly dependent, then at least one of the vectors of the set can be expressed as a linear combination of the remaining vectors.

Conversely, if one of the vectors of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linear combination of the remaining others, the set is linearly dependent.

Proof: Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly dependent, there exist $c_1, c_2, \ldots, c_n$ in F, not all zero, s.t

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0.$$

Let $c_j \neq 0$, then $c_j^{-1} \in F$ and $c_j^{-1}c_j = 1$, the identity element of F.

Now, $c_j\alpha_j = -c_1\alpha_1 - c_2\alpha_2 - \cdots - c_{j-1}\alpha_{j-1} - c_{j+1}\alpha_{j+1} - \cdots - c_n\alpha_n$.

$$\therefore c_j^{-1}(c_j\alpha_j) = c_j^{-1}(-c_1\alpha_1 - c_2\alpha_2 - \cdots - c_{j-1}\alpha_{j-1} - c_{j+1}\alpha_{j+1} - \cdots - c_n\alpha_n)$$

$$\text{or } (c_j^{-1}c_j)\alpha_j = (-c_j^{-1}c_1)\alpha_1 + (-c_j^{-1}c_2)\alpha_2 + \cdots + (-c_j^{-1}c_{j-1})\alpha_{j-1} + (-c_j^{-1}c_{j+1})\alpha_{j+1} + \cdots + (-c_j^{-1}c_n)\alpha_n$$

$$\text{or } \alpha_j = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_{j-1}\alpha_{j-1} + d_{j+1}\alpha_{j+1} + \cdots + d_n\alpha_n.$$

Hence α_j is the linear combination of the other vectors.

Conversely, let one of the vectors α_j is a linear combination of the other vectors of the set, then

$$\alpha_j = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_{j-1}\alpha_{j-1} + r_{j+1}\alpha_{j+1} + \cdots + r_n\alpha_n.$$

for some scalars $r_i \in F$.

$$\therefore r_1\alpha_1 + r_2\alpha_2 + \cdots + r_{j-1}\alpha_{j-1} + (-1)\alpha_j + r_{j+1}\alpha_{j+1} + \cdots + r_n\alpha_n = 0.$$

Since one of the scalars $r_1, r_2, \ldots, r_{j-1}, -1, r_{j+1}, \ldots, r_n$ is non-zero, the set $\alpha_1, \alpha_2, \ldots, \alpha_n$ is linearly dependent.

Deletion Theorem.

If a non null vector space V_F be spanned by a linearly dependent set $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$. Then V can also be spanned by a suitable proper subset of S.

Proof.

Since S is linearly dependent, let $\alpha_j \in S$ and

$$\alpha_j = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_{j-1}\alpha_{j-1} + c_{j+1}\alpha_{j+1} + \cdots + c_n\alpha_n$$

for some scalars $c_i \in F$. $i = 1, 2 \ldots j-1, j+1, \ldots n$

Let $T = \{\alpha_1, \alpha_2 \ldots \alpha_{j-1}, \alpha_{j+1} \ldots \alpha_n\}$.

Then $T \in S$

Hence $L(T) \subset L(S)$.

Also $L(S) = V$.

Again elements of S is linear combination of elements of T.

$\therefore L(S) \subset L(T)$.

Hence $L(S) = L(T)$.

$\therefore L(T) = V \quad (\because L(S) = V)$

Thus V is spanned by a proper subset of S.

Ex. Let $\alpha_1 = (1, 2, 0)$, $\alpha_2 = (3, -1, 1)$, $\alpha_3 = (4, 1, 1)$

Apply deletion theorem to show that a proper subset of $S = \{\alpha_1, \alpha_2, \alpha_3\}$ spans V.

Sol^n: Let $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$

$\therefore c_1(1, 2, 0) + c_2(3, -1, 1) + c_3(4, 1, 1) = (0, 0, 0)$

$\therefore c_1 + 3c_2 + 4c_3 = 0$

$2c_1 - c_2 + c_3 = 0$

$c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$.

$\therefore 2c_1 + 2c_3 = 0 \Rightarrow c_1 = -c_3 \Rightarrow c_1 = c_2$

$\therefore c_1 = c_2 = -c_3$.

$\therefore$ S is linearly dependent.

Taking $c_1 = 1$, we have $c_2 = 1$, $c_3 = -1$

$\therefore \quad \alpha_1 + \alpha_2 = \alpha_3$.

Hence by deletion theorem α_3 can be deleted from S & $L\{\alpha_1, \alpha_2\} = L\{\alpha_1, \alpha_2, \alpha_3\}$.

Basis & Dimension.

A subset B of a vector space V_F is called a basis of V if

i) B is linearly independent and

ii) B spans V.

Defn. A vector space V_F is said to be finitely generated (or finite Dimensional) if it has a finite ~~set~~ subset which spans V.

Ex. The set $S = \{\epsilon_1 = (1,0), \epsilon_2 = (0,1)\}$ is a basis of R^2.

$S = \{\epsilon_1, \epsilon_2\}$ is linearly independent since

$c_1 \epsilon_1 + c_2 \epsilon_2 = \theta \Rightarrow$

$c_1 = 0, \; c_2 = 0$.

Also let $\xi \in R^2$ be any vector.

Then $\xi = (a, b)$ can be expressed as

$(a,b) = a(1,0) + b(0,1)$.

$\quad = a\epsilon_1 + b\epsilon_1$

$\therefore \xi \in L(S)$. $\therefore R^2 \subset L(S)$.

Also $S \subset R^2$ & $L(S)$ being the smallest subspace of R^2 containing S, $L(S) \subset R^2$.

$\therefore \quad L(S) = R^2$

$\therefore$ S is a basis of R^2 since it satisfies both the conditions

Theorem:

If $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis of a vector space V_F and $\beta \in V$ be a non-zero vector & $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$, for some $c_i \in F$ then if $c_j \neq 0$, $\{\alpha_1, \alpha_2, \ldots \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots \alpha_n\}$ is a new basis of V.

Theorem.

If $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis of finite dimensional vector space V_F, then any linearly independent set of vectors in V contains at most n vectors.

Theorem:

Any two bases of a vector space V_F of finite dimensional have the same number of vectors.

Defn.

The number of vectors in a basis of a vector space V_F is said dimension of V and is denoted by $\dim(V)$.

Eg. 1. The dimension of the vector space R^2 is 2 since $E = \{e_1 = (1,0), e_2 = (0,1)\}$ is a basis.

2. The dimension of the vector space R^3 is 3 since $E = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is a basis.

3. The dimension of $R_{m \times n}$ of all $m \times n$ real matrices is mn since the set $\{E_{11}, E_{12} \cdots E_{mn}\}$ where E_{ij} is a $m \times n$ matrix having 1 at the (i,j) position and zero elsewhere is a basis.

Th. Let the vector space V is of dimension n. Then any linearly independent set of n vectors is a basis of V.

Th. Let the vector space V_F is of dimension n. Then any subset of n vectors of V that generates V is a basis of V.

Ex. Find a basis of R^3 that contains $(1,2,1)$ & $(3,6,2)$.

Sol. R^3 is a vector space with its standard basis as $\epsilon_1 = (1,0,0)$, $\epsilon_2 = (0,1,0)$, $\epsilon_3 = (0,0,1)$ so that any vector $\alpha = (a,b,c) \in R^3$ can be expressed as $\alpha = (a,b,c) = a\epsilon_1 + b\epsilon_2 + c\epsilon_3$.

Let $\beta = (1,2,1)$ & $\gamma = (3,6,2)$.

Now, $\beta = (1,2,1) = 1\epsilon_1 + 2\epsilon_2 + 1\epsilon_3$

Since the coefficients of $\epsilon_1, \epsilon_2, \epsilon_3$ are non-zero in the above expression so by replacement theorem β can replace any one of $\epsilon_1, \epsilon_2, \epsilon_3$. Suppose ϵ_1 is replaced by β in the basis. Then $\{\beta, \epsilon_2, \epsilon_3\}$ is the new basis of R^3.

Again let $\gamma = c_1\beta + c_2\epsilon_2 + c_3\epsilon_3$.

$\therefore (3,6,2) = c_1(1,2,1) + c_2(0,1,0) + c_3(0,0,1)$.

$\therefore c_1 = 3, \quad 2c_1 + c_2 = 6, \quad c_1 + c_3 = 2$

$\Rightarrow c_2 = 0 \qquad \Rightarrow c_3 = -1$.

Since $c_3 \neq 0$ so $\{\beta, \epsilon_2, \gamma\}$ is the new basis of R^3 by Replacement theorem.